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Induced Representations: A Retrospect

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Abstract
This work delves into the intricate realm of induced representations, offering an exhaustive examination of three pivotal theorems in the domain of induced representations pertaining to locally compact groups. These three theorems, specifically, are the theorem on induction in stages, the imprimitivity theorem and the intertwining number theorem. These theorems are central to understanding how representations are constructed, related, and interwoven within the framework of group theory.

Keywords: Induced representation, imprimitivity, intertwining.

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1. Introduction

Let G be a locally compact group. Let μ be a left Haar measure on G. The following definitions serve as a bedrock for the sequel [1–3]

Definition 1.1. Let μ_x be a Borel measure on G. Then for each x, there exists a number $\Delta(x) \geq 0$ such that $\mu_x = \Delta(x)\mu$. The function $\Delta: G \to \mathbb{R}$ is called the modular function of G. If $\Delta = 1$, then G is said to be unimodular.

Remark 1.2. From the definition above (1.1), we deduce the following:

(a) Δ is continuous, and
(b) $\Delta(x,y) = \Delta(x)\Delta(y)$ $for\ all\ x, y \in G$.

Definition 1.3. Let f be a continuous function on a topological space X. The support of f written $supp(f)$ is the closure of $\{x \in \mathrm{X}: f(x) \neq 0\}$.

$C_o(\mathrm{X})$ denotes the set of continuous functions on X with compact support.

Now, let H be a closed subgroup of G.

Let right invariant measure da on G and $d\xi$ on H with corresponding modular functions Δ and δ respectively.

In particular, for any integrable function f on G,

$$\int_G f(ba)da = \Delta(b^{-1})\int_G f(a)da$$

$$\int_G f(a^{-1})\Delta(a^{-1})da = \int_G f(a)da,$$ and similarly for δ.

In addition, let $\rho(\xi) = \delta(\xi)\Delta(\xi)^{-1}$ for $\xi \in H$.

Let $\pi: G \to M$ be the canonical projection onto the space of left cosets $M = H/G$ and let τ be the averaging map $(\tau\varphi)(\pi(x)) = \int \varphi(\xi x) d\xi \ (x \in G)$ for $\varphi \in C_0(G)$.

Suppose L is a continuous unitary representation of H on a Hilbert space $\mathcal{V}$ and let F^* be the set of functions

$f: G \to \mathcal{V}$ satisfying:

(a) $f(\xi_a) = \rho(\xi)^{\frac{1}{2}} L(\xi) f(a)$ that is,

$f(\xi_a) = \delta(\xi)^{\frac{1}{2}} \Delta(\xi)^{-\frac{1}{2}} L(\xi) f(a) \ (\delta \in H, a \in G)$

(b) $\|f(a)\|^2$ is locally integrable, that is,
$\int \|f(a)\|_{\mathcal{V}}^2 \ da < \infty \ where \ \|\cdot\|_{\mathcal{V}}$ is a norm in $\mathcal{V}$

(c) f is strongly measurable, that is, $\|f(a)\|$is measurable and $< f(a), v > (v \in \mathcal{V})$ and for every compact subset $\mathcal{V}_o \ of \ \mathcal{V}$such that $f(a) \epsilon \overline{\mathcal{V}_0}$(almost everywhere in K).

Next, we show that F^* defines an inner product space, with the inner product:
$< f_1, f_2 >= \int_G < f_1(a), f_2(a) > da$.

The proof is given in five points:

(i) $$< f_1, f_2 >= \int_G < f_1(a), f_2(a) > da.$$
$$= \int_G \| f_1(a) \|^2 da \geq 0.$$

(ii) Here we show that $< f_1, f_1 > = 0$ if and only if $f_1 = 0$.
Let$< f_1, f_1 > = 0$. Then $\int_G \| f_1 \|^2 da = 0$ .That is,$\|f_1(a)\| = 0$, and so,$f_1(a) = 0$, for all $a \in G$.
Therefore $f_1 = 0$.
Conversely, if $f_1 = 0$,then $f_1(a) = 0$ for all $a \in G$, that is, $\int_G \| f_1(a) \|^2 da = 0$ and.
Thus,$< f_1, f_1 > = 0$.

(iii) Let $f_1, f_2 \ and \ f_3$ be functions in F^*
$$< f_1 + f_2 . f_3 >= \int_G < (f_1 + f_2)(a), f_3(a) > da$$
$$= \int < f_1(a) + f_2(a), f_3(a) > da.$$

Since $\mathcal{V}$ is a Hilbert space,
$< f_1(a) + f_2(a), f_3(a) > =< f_1(a), f_3(a) > +< f_2(a), f_3(a) >$.
Therefore,
$$< f_1 + f_2, f_3 >= \int_G < f_1(a), f_3(a) > da + \int_G < f_2(a), f_3(a) > da$$
And so, $< f_1 + f_2, f_3 > =< f_1, f_3 > +< f_2, f_3 >$

(iv) Let $\lambda \in \mathbb{K}$ and $f_1, f_2 \in F^*$
$$< \lambda f_1, f_2 >= \int_G < (\lambda f_1)(a), f_2(a) > da$$
$$= \int_G < \lambda f_1(a), f_2(a) > da$$

$$= \int_G \lambda < f_1(a), f_2(a) > da \text{ (since } \mathcal{V} \text{ is a Hilbert space)}$$
$$= \lambda \int_G < f_1(a), f_2(a) > da$$
That is, $< \lambda f_1, f_2 > = \lambda < f_1, f_2 >$.
(v) Let $f_1, f_2 \in F^*$
$$< f_1, f_2 > = \int_G < f_1(a), f_2(a) > da$$
$$= \int_G \overline{< f_2(a), f_1(a) >} da$$
$$= \overline{\int_G < f_2(a), f_1(a) > da}$$
That is, $< f_1, f_2 > = \overline{< f_2, f_1 >}$.
Hence F^* defines an inner product space, say $\mathcal{H} \subset F^*$
Lemma 1.4. For $f \in F^*$ and $\varphi \in C_o(G)$, $\mu_{f_1 f} : \tau\varphi \mapsto \int_G \| f(a) \|^2 \varphi(a) da$ is a Radon measure on M, that is, a continuous linear functional on $C_o(M)$. Then
$$\int_G \| f(a) \|^2 \varphi(a) da = \int_M (\tau\varphi)(p) d_{\mu_f}(p)$$
Now, for $f \in F^*$, Let $\|f\|^2 = \mu_{f,f}(M)$ and let $H^L = \{f \in F^* : \|f\| < \infty\}/\{f \in F^* : \|f\| = 0\}$.
Then $(U^L(g)f)(a) = f(a \cdot g) \quad (a, g \in G)$ defines a continuous unitary representation U^L in H^L of $G. U^L$ is called the induced representation of L from H to G [4– 7]. In this section, we discuss the theorem on induce in stages. However, we first introduce a function called $\epsilon - map$ and elaborate on its properties.

2. Induction in stages

In this section, we discuss the theorem on inducing in stages. However, we first introduce a function called $\epsilon - map$ and elaborate on its properties.
Let $f \in C_o(G)$ and $v \epsilon \mathcal{V}$.
We form
$$\in (f, v)x = \int_H \delta(\xi)^{-\frac{1}{2}} \Delta(\xi)^{\frac{1}{2}} f(\xi x) L(\xi)^{-1} v d\xi$$
$\epsilon(f, v)$ has its support contained in HK if the support of f is K.
Let F_o be the subset of F^* consisting of functions that are continuous with compact support modulo H.
Thus $F_o \subseteq \{f \in F^*: \|f\| < \infty\}$. Then $\epsilon(f, v) \in F_o$.
Definition 2.1. We say that a subset S of a topological vector space $\mathcal{V}$ is total if the linear manifold spanned by S is dense in $\mathcal{V}$.

In the following, we give two key properties of the $\epsilon - map$ [8]:
Lemma 2.2. (a) If K is the support of f, then $\|\epsilon(f, v)\| \leq \lambda_K \|f\|_G \|v\|$
(b) If $\mathfrak{D}$ is total in a Hilbert space $\mathcal{V}$, then $\epsilon(C_o(G) \times \mathfrak{D})$ is total in $\mathcal{H}$.
We end this section by giving a proof of the theorem through induction in stages following Blattner's version [8–12].
Theorem 2.3. Let H_1 and H_2 be closed subgroups of G with $H_1 \subseteq H_2$. Let L be the unitary representation of H_1 on a Hilbert space $\mathcal{V}$ and denote the inductions of L to H_1 and G by M and U respectively. Then U is unitary equivalent to U^M.

In some other notations, we write:

$$ind\ (G, H_1, L) \sim ind(G, H_2\,, ind\ (H_2, H_1, L))$$

or

$$ind_{H_1}^{G} \sim ind_{H_2}^{G}\, ind_{H_1}^{H_2}.$$

Proof. Suppose δ_1, δ_2 *and* Δ are the modular functions for H_1, H_2 and G respectively. Let $F^{(1)}, F^{(2)}$ and F be the spaces for the inductions from H_1 *to* H_2 , H_2 to G and H_1 to G respectively.

Let $f \in F_o$ with support in the compact set H_1K.

For $\eta \in H_2, x \epsilon G$, let $\hat{f}(\eta, x) = \delta_2(\eta)^{-\frac{1}{2}}\Delta(\eta)^{\frac{1}{2}}f(\eta)$ (2.1)

Let x be fixed.

Then
$$\begin{aligned}\hat{f}(\xi\eta, x) &= \delta_2(\xi\eta)^{-\frac{1}{2}}\Delta(\xi\eta)^{\frac{1}{2}}f(\xi\eta x)\\ &= \delta_2(\xi)^{-\frac{1}{2}}\delta_2(\eta)^{-\frac{1}{2}}\Delta(\xi)^{\frac{1}{2}}\Delta(\eta)^{\frac{1}{2}}\Delta(\xi)^{-\frac{1}{2}}\delta_1(\xi)^{\frac{1}{2}}L(\xi)f(\eta x)\\ &= \delta_2(\xi)^{-\frac{1}{2}}\delta_1(\xi)^{\frac{1}{2}}\delta_2(\eta)^{-\frac{1}{2}}\Delta(\eta)^{\frac{1}{2}}L(\xi)f(\eta x)\\ &= \delta_2(\xi)^{-\frac{1}{2}}\delta_1(\xi)^{\frac{1}{2}}\mathrm{L}(\xi)\hat{f}(\eta, x) \quad \text{(from (2.1))}\\ &\qquad (\xi \in H_1, \eta \in H_2)\end{aligned}$$

Furthermore, $\hat{f}(\cdot, x)$ is a continuous with support in $H_1\left(Kx^{-1} \cap K_2\right)$. So $\hat{f}(\cdot, x)$ belongs to $F_o^{(1)}$, that we denote by $\hat{f}(x)$.

Now, let $\eta, \xi \in H_2, x \in G$,

$\hat{f}(\eta, \zeta x) = \delta_2(\eta)^{-\frac{1}{2}}\Delta(\eta)^{\frac{1}{2}}f(\eta\zeta x)$ (from (2.1))

$= \delta_2(\eta)^{-\frac{1}{2}}\Delta(\eta)^{\frac{1}{2}}\Delta(\eta\zeta)^{-\frac{1}{2}}\delta_2(\eta\zeta)^{\frac{1}{2}}L(\eta\zeta)f(x)$ (from the definition of F^*)

$= \delta_2(\eta)^{-\frac{1}{2}}\Delta(\eta)^{\frac{1}{2}}\Delta(\eta)^{-\frac{1}{2}}\Delta(\zeta)^{-\frac{1}{2}}\delta_2(\eta)^{\frac{1}{2}}\delta_2(\zeta)^{\frac{1}{2}}L(\eta\zeta)f(x)$ (from the definition of a modular function)

$$\begin{aligned}&= \delta_2(\zeta)^{\frac{1}{2}}\Delta(\zeta)^{-\frac{1}{2}}L(\eta\zeta)f(x)\\ &= \delta_2(\zeta)^{\frac{1}{2}}\Delta(\zeta)^{-\frac{1}{2}}\Delta(\eta\zeta)^{\frac{1}{2}}\delta_2(\eta\zeta)^{-\frac{1}{2}}f(\eta\zeta x)\\ &= \delta_2(\zeta)^{\frac{1}{2}}\Delta(\zeta)^{-\frac{1}{2}}\hat{f}(\eta\zeta x) \quad \text{(from (2.1))}\end{aligned}$$

Hence,

$\hat{f}(\zeta x) = \delta_2(\zeta)^{\frac{1}{2}}\Delta(\zeta)^{-\frac{1}{2}}M_\zeta f(x)$.

The support of $\hat{f}(\cdot)$ is in H_2K .

To show continuity, suppose N is a compact neighborhood of e in G and choose $h \in C_o(G)$ so that $\int_{H_1} h(\xi x)d\xi = 1$ on H_1KN .

Then

$\int_{H_1} h(\xi\eta x)d\xi = 1$ for $\eta \in H_1(KNx^{-1} \cap H_2)$

Thus, $\left\| \hat{f}(x) - \hat{f}(y) \right\|^2 = \int_{H_2} h(\eta x)\left\| \hat{f}(\eta, x) - \hat{f}(\eta, y) \right\|^2 d\eta$

$$= \int_{H_2} h(\eta x)\left\| \delta_2(\eta)^{-\frac{1}{2}}\Delta(\eta)^{\frac{1}{2}}f(\eta x) - \delta_2(\eta)^{-\frac{1}{2}}\Delta(\eta)^{\frac{1}{2}}f(\eta y) \right\|^2 d\eta$$

$$= \int_{H_2} h(\eta x)\left(\delta_2(\eta)^{-\frac{1}{2}}\Delta(\eta)^{\frac{1}{2}}\right)^2 \| f(\eta x) - f(\eta y)\|^2 d\eta$$

$$= \int_{H_2} \delta_2(\eta)^{-1}\Delta(\eta)h(\eta x)\| f(\eta x) - f(\eta y)\|^2 d\eta$$

whenever $y^{-1}x \in N$.

Clearly $\hat{f}$ is here uniformly continuous on compact sets, and so it is continuous. Then $\hat{f}(\cdot)$ belongs to $F_o^{(2)}$, and will be abbreviated as $\hat{f}$.

Now, $\left\| \hat{f}(x)\right\|^2 = \int_{H_2} \delta_2(\eta)^{-1}\Delta(\eta)h(\eta x)\| f(\eta x)\|^2 d\eta$.

Let $k \in C_o(G)$ such that $\int_{H_2} k(\eta x)d\eta = 1 \text{ on } H_2K$.

Then due to the Fubini's theorem and the Haar measure's preservation under group action, we have:

$$\left\|\hat{f}\right\|^2 = \int_G k(x)\left\|\hat{f}(x)\right\|^2 dx$$

$$= \int_G \int_{H_2} k(x)\delta_2(\eta)^{-1}\Delta(\eta)h(\eta x)\| f(\eta x)\|^2 d\eta dx$$

$$= \int_{H_2} \int_G k(x)\Delta(\eta)\delta_2(\eta)^{-1}h(\eta x)\| f(\eta x)\|^2 dx d\eta$$

$$= \int_{H_2} \int_G k(\eta^{-1}x)\delta_2(\eta)^{-1}h(x)\| f(x)\|^2 dx d\eta$$

$$= \int_G h(x)\| f(x)\|^2 \left[\int_{H_2} k(\eta x)d\eta\right] dx$$

$$= \int_G h(x)\| f(x)\|^2 dx \,(\text{since} \int_{H_2} k(\eta x)d\eta = 1)$$

$$= \| f\|^2 \qquad \text{(by the choice of } h \text{ and } k).$$

Hence $f \to \hat{f}$ is an isometry of F_o onto $F_o^{(2)}$.

Next we prove that the image of F_o is dense in $F_o^{(2)}$.

Let $g \in C_o(H_2)$, $h \in C_o(G)$, $v \in \mathcal{V}$ and take

$$k(x) = \int_{H_2} \delta_2(\zeta)^{-\frac{1}{2}}\Delta(\zeta)^{\frac{1}{2}} g(\zeta)^{-1}h(\zeta x)d\zeta \in C_o(G).$$

Then

$$\in (k,v)(x) = \int_{H_1} \delta_1(\xi)^{-\frac{1}{2}}\Delta(\xi)^{\frac{1}{2}} k(\xi x)\, L(\xi)^{-1} v d\xi$$

$$= \int_{H_1}\int_{H_2} \delta_1(\xi)^{-\frac{1}{2}}\Delta(\xi)^{\frac{1}{2}}\delta_2(\zeta)^{-\frac{1}{2}}\Delta(\zeta)^{\frac{1}{2}} g(\zeta)^{-1}h(\zeta\xi x)\, \mathrm{L}(\xi)^{-1} v d\zeta d\xi$$

$$= \int_{H_1}\int_{H_2} \delta_1(\xi)^{-\frac{1}{2}}\delta_2(\zeta)^{-\frac{1}{2}}\Delta(\zeta\xi)^{\frac{1}{2}} g(\zeta)^{-1}h(\zeta\xi x)\, \mathrm{L}(\xi)^{-1} v d\zeta d\xi$$

so that,

$$\epsilon(k,v)^\wedge(\eta,x) = \int_{H_1}\int_{H_2} \delta_1(\xi)^{-\frac{1}{2}}\delta_2(\zeta\eta)^{-\frac{1}{2}}\Delta(\zeta\xi\eta)^{\frac{1}{2}} g(\zeta)^{-1}h(\zeta\xi\eta x)\, \mathrm{L}(\xi)^{-1} v d\zeta d\xi$$

$$= \int_{H_1}\int_{H_2} \delta_1(\xi)^{-\frac{1}{2}} \delta_2(\zeta\xi^{-1})^{-\frac{1}{2}} \Delta(\zeta)^{\frac{1}{2}} g(\xi\eta\zeta^{-1}) h(\zeta x) \mathrm{L}(\xi)^{-1} v d\zeta d\xi$$

$$= \int_{H_2} \delta_2(\zeta)^{-\frac{1}{2}} \Delta(\zeta)^{\frac{1}{2}} h(\zeta x) \left[\int_{H_1} \delta_1(\xi)^{-\frac{1}{2}} \delta_2(\xi)^{\frac{1}{2}} g(\xi\eta\zeta^{-1}) L(\xi)^{-1} v d\xi \right] d\zeta$$

Take $\epsilon_1\ and\ \epsilon_2$ as the $\epsilon - maps$ for the inductions from $H_1\ to\ H_2$ and $H_2\ to\ G$ respectively.

Then we have $\epsilon(k, v)^\wedge = \epsilon_2(h, \epsilon_1(g, v))$.

By Lemma 2.2, the set $\epsilon_2(C_o(G)) \times \epsilon_1(C_o(H_2) \times V)$ is total in $F^{(2)}$. This shows that the image of F_o in $F_o^{(2)}$ is dense. Therefore, the map $f \to \hat{f}$ can be extended to a unitary map of F onto $F^{(2)}$. This proves the desired equivalence.

■

3. The intertwining number theorem

In this section, we first state some pre-requisite facts needed for a standard proof of the intertwining number theorem.

Definition 3.1. A one-parameter subgroup of a Lie group G is an analytic homomorphism, say $\theta$from $\mathbb{R}\ to\ G$.

Now, let G be a Lie group and V a unitary representation of G on the Hilbert space $\mathcal{K}$.Let $X \in g$, the left-invariant Lie algebra of $G$and let $x(\cdot)$ be the one-parameter subgroup of G such that

$(Xf)(y) = D_t f(yx(t))\big|_{t=0}$ for all $f \in C_o^\infty(G)$.

$dV(x)$ denotes the skew-adjoint infinitesimal operator generating the one-parameter unitary group $V_{x(\cdot)}$ in $\mathcal{K}$.

Let $\mathcal{K}_\infty$ be the largest submanifold of $\mathcal{K}$ contained in $\cap \left[dom\big(dV(x)\big): X \in g\right]$ and it is invariant under $dV(g)$.

Since $dV(X)V_y = V_y dV(ad_{y^{-1}} X)$,$X \in g, y \in G, \mathcal{K}_\infty$is V-Invariant.

Denote the restriction of dV to $\mathcal{K}_\infty$ by ∂V.

The following two lemmas are extracted from the work of Blattner [8]:

Lemma 3.2. Suppose $V^1$and $V^2$are unitary representations of G on the Hilbert spaces $\mathcal{K}^1$and $\mathcal{K}^2$ respectively. Let $A \in \Re(V^1, V^2)$, the set of operators intertwining $V^1$and V^2. Then $A\mathcal{K}_\infty^1 \subseteq \mathcal{K}_\infty^2$.Furthermore, $A\partial V^1(X) \subseteq \partial V^2(X)A$ for $X \in \mathcal{L}$ where $\mathcal{L}$ is the enveloping algebra of the complexification of g.

Lemma 3.3. Suppose $f \in C_o^\infty(G)$,$v \in \mathcal{V}$. Then $\epsilon(f, v)\epsilon\mathcal{H}_\infty$. Furthermore, $\partial U^L(X)\epsilon(f, v) = \epsilon(Xf, v)$ for each $X \in \mathcal{L}$.

Definition 3.4.

(a) An elliptic element is an element which is regarded as a left-invariant (analytic) linear differential operator.

(b) $\mathcal{H}_\infty$ denotes the domain of all operators of the differential representation ∂U^L of the enveloping algebra $\mathcal{L}$ of the Lie algebra g of G.

Theorem 3.5.
Let $\mathcal{V}$ be infinite-dimensional. Then $\mathcal{H}_\infty \subseteq C(G;\mathcal{V})$. Furthermore, suppose that X_o is an elliptic element of $\mathcal{L}$ of order $m>\frac{n}{2}$ where $n=\dim G/H$. Then for all compact subsets K of G, there exists a constant c_K such that $\|g\|_K \subseteq c_K\left(\left\|\partial U^L(X_o)g\right\|+\|g\|\right)$ for all $g \in \mathcal{H}_\infty$.
Now, consider two closed subgroups $H_1\ and\ H_2$ of a Lie group G with modular functions δ_1 and δ_2 respectively. Let $L^{(1)}$ be a unitary representation of H_i on the Hilbert space $\mathcal{V}_i, i = 1,2$. $U^{L^{(i)}}$ operates on $\mathcal{H}^{(i)}$. $\mathcal{L}(\mathcal{V}_1,\mathcal{V}_2)$ is the space of all bounded linear operators of $\mathcal{V}_1\ into\ \mathcal{V}_2$ endowed with the bounded convergence topology.
We assume $dim\mathcal{V}_2 < \infty$.
For every $A \in \Re(U^{L^{(1)}}, U^{L^{(2)}})$, we define a linear map r_A from $C_o^\infty(G)$ to the set of linear maps of $\mathcal{V}_1\ into\ \mathcal{V}_2$ as follows: for every $f \in C_o^\infty(G)$ and $v \in \mathcal{V}_1, set\ r_A(f)v = (A\epsilon(f,v))(e)$.
For $(\xi_1,\xi_2) \in H_1 \times H_2$ and any function f on G, we set $(\rho\epsilon_1,\epsilon_2 f) = f(\xi_1^{-1}x\xi_2),\ x \in G.$
Now, we can state the main result of this section:
Theorem 3.6.(Intertwining Number Theorem) [9,13,14]
Let X_o be an elliptic element of $\mathcal{L}$ of order greater than $\frac{1}{2}\dim\left(G/H_2\right)$.For $f \in C_o^\infty(G)$,set $\|f\|_{X_o} = \|X_o f\|_G + \|f\|_G$.
For each relatively compact open set $O\ of\ G$, endow $C_o^\infty(O)$ with the topology induced by $\|\cdot\|_{X_o}$; endow $C_o^\infty(o)$ with the corresponding inductive limit topology. Suppose $\mathcal{M}$ is the subspace of maps
$z \in \mathcal{L}(C_o^\infty(G);\mathcal{L}(\mathcal{V}_1;\mathcal{V}_2))$ such that $z\left(\rho_{\xi_1,\xi_2} f\right)= \delta_1(\xi_1)^{\frac{1}{2}}\delta_2(\xi_2)^{\frac{1}{2}}\Delta(\xi_1\xi_2^{-1})^{\frac{1}{2}}L^{(2)}(\xi_2)\mathrm{z}(f)L^{(1)}(\xi_1)^{-1}$ for all $(\xi_1,\xi_2) \in H_1\times H_2$ and all $f \in C_o^\infty(G)$.
Then the map $A \to r_A$ is a faithful linear map of $\Re\left(U^{L^{(1)}}, U^{L^{(2)}}\right)$ into $\mathcal{M}$.

Proof. Let $v \in \mathcal{V}_1$, $A \in \Re\left(U^{L^{(1)}}, U^{L^{(2)}}\right)$, $f \in C_o^\infty(O)$ and O a relatively compact open subset of G.
By Theorem 3.5.,

$$\|r_A(f)v\| \le C_{\{e\}}\left(\left\|\partial U^{L^{(2)}}(X_o)A \in (f,v)\right\| + \|A \in (f,v)\|\right) \tag{3.1}$$

By Lemma 3.2., $\partial U^{L^{(2)}}(X_o)\ A \in (f,v) = A\partial U^{L^{(1)}}(X_o) \in (f,v)$.
Moreover by Lemma 3.3, we obtain $\partial U^{L^{(1)}}(X_o) \in (f,v) =\in (X_o f,v)$,and so,
$\partial U^{L^{(2)}}(X_o) \in (f,v) = A \in (X_o f,v)$.

$$\begin{aligned}\text{Thus, } \|r_A(f)v\| &\le C_{\{e\}}(\|A \in (X_o f,v)\| + \|A \in (f,v)\|)\\ &= C_{\{e\}}(\|A\|\|\in (X_o f,v)\| + \|A\|\|\in (f,v)\|)\\ &= C_{\{e\}}\|A\|(\|\in (X_o f,v)\| + \|\in (f,v)\|)\end{aligned}$$

By Lemma 2.2, $\|\in(X_o f,v)\| \leq \lambda_{K_1}\|X_o f\|_G\|v\|$ and
$\|\in(f,v)\| \leq \lambda_{K_2}\|f\|_G\|v\|$.
Therefore (3.1) becomes

$$\|r_A(f)v\| \leq C_{\{e\}}\|A\|\left(\lambda_{K_1}\|X_o f\|_G\|v\| + \lambda_{K_2}\|f\|_G\|v\|\right)$$
$$\leq C_{\{e\}}\|A\|\lambda_{K'}\|v\|(\|X_o f\|_G + \|f\|_G)$$

where $\lambda_{K'} = \max\{\lambda_{K_1}, \lambda_{K_2}\}$.
Moreover, by hypothesis, $\|X_o f\|_G + \|f\|_G = \|f\|_{X_o}$.
Hence, $\|r_A(f)v\| \leq C_{\{e\}}\|A\|\lambda_{K'}\|v\|\|f\|_{X_o}$ and so, $r_A \in L\left(C_o^\infty(G); L(\mathcal{V}_1, \mathcal{V}_2)\right)$.
Now, let $r_A = 0$. Let $f \in C_o^\infty(G)$ and $v \in \mathcal{V}_1$.
For every $x \in G$, we have:
$\left(A \in (f,v)\right)(x) = \left({U_x}^{L^{(2)}} A \in (f,v)\right)(e)$ but ${U_x}^{L^{(2)}}A = A{U_x}^{L^{(1)}}$ $\Big($$since\ A \in \mathcal{R}(U^{L^{(1)}}, U^{L^{(2)}})$$\Big)$ so $\left(A \in (f,v)\right)(x) = \left(A{U_x}^{L^{(1)}} \in (f,v)\right)(e)$

$$= \left(A \in (R_x f, v)\right)(e)$$

However, $\left(A \in (f,v)\right)(e) = r_A(f)v$ for all $f \in C_o^\infty(G)$ and $v \in \mathcal{V}_1$.
Thus, $\left(A \in (f,v)\right)(x) = r_A(R_x f)(v)$ (3.2)
Since $r_A = 0$, $\left(A \in (f,v)\right)(e) = 0$.
Then by Lemma 2.2, $A = 0$ on $\in(C_o^\infty(G) \times \mathcal{V}_1)$, a total subset of $\mathcal{H}^{(1)}$ and so r_A is one-to-one, that is faithful($since$ $\ker(r_A) = \{0\}$).
Next, we prove that $r_A \in \mathcal{M}$.
Let $f \in C_o^\infty(G)$, $v \in \mathcal{V}_1$ and $(\xi_1, \xi_2) \in H_1 \times H_2$.
Then $r_A\left(\rho_{\xi_1,\xi_2} f\right)(v) = r_A\left(R_{\xi_1}\rho_{\xi_2,e} f\right)v$

$$= \left(A \in (\rho_{\xi_1,e} f, v)\right)(\xi_2) \quad \text{(from (3.2))}$$
$$= \delta_2(\xi_2)^{\frac{1}{2}}\Delta(\xi_2)^{-\frac{1}{2}}L^{(2)}(\xi_2)r_A\left(\rho_{\xi_1,e} f\right) \text{by property (a) for functions of } F^*$$

Additionally,

$$\in\left(\rho_{\xi_1,e} f, v\right)(x) = \int_{H_1}\delta_1(\xi)^{-\frac{1}{2}}\Delta(\xi)^{\frac{1}{2}} f\left(\xi_1^{-1}\xi x\right)L^{(1)}(\xi)^{-1}v\,d\xi$$
$$= \int_{H_1}\delta_1(\xi_1\xi)^{-\frac{1}{2}}\Delta(\xi_1\xi)^{\frac{1}{2}} f(\xi x)L^{(1)}(\xi^{-1}\xi_1^{-1})v\,\delta_1(\xi_1)\,d\xi$$
$$= \delta_1(\xi_1)^{\frac{1}{2}}\Delta(\xi_1)^{\frac{1}{2}}\left(f, L^{(1)}(\xi_1^{-1})v\right)(x)$$

Then $r_A \in \mathcal{M}$.

■

Remarks 3.7. (1) If $V_1\ and\ V_2$ are representations of a Lie group G, then $dim\mathcal{R}(V_1, V_2)$ is called the intertwining number of $V_1\ and\ V_2$ and is denoted by $I(V_1, V_2)$.
(2) $I\left(U^{L^{(1)}}, U^{L^{(2)}}\right) \leq \dim \mathcal{M}$.

4. Imprimitivity

In this section, we will delve into the details of the imprimitivity theorem following the version of Bent Orsted [15]. This method is different from the classical proofs found in some literatures [10,16–21]. We equally note that if we let$(P^L(\Psi)f)(a) = \Psi\big(\pi(a)\big)f(a)(f \in H^L, \Psi \in C_o(M))$, then the pair (U^L, P^L) is called an induced system of imprimitivity.

Theorem 4.1.(Imprimitivity Theorem)

Let U be a continuous unitary representation of $G\ in\ \mathcal{V}$. Let $P: C_o(M) \to \mathcal{L}(\mathcal{V})$ be a homomorphism with$P(C_o(M))\mathcal{V}$ dense in $\mathcal{V}$ and

$$U(g)P(\Psi)U(g)^{-1} = P(R(g)\Psi)(g \in G, \Psi \in C_o(M))$$

where $(R(g)\Psi)(\pi(x)) = \Psi(\pi(xg))$.

Then there is a unique continuous unitary representation L(up to unitary equivalence) of a closed subgroup $H\ of\ G$ in a Hilbert space $\mathcal{V}^1$ such that $(U, P)\sim(U^L, P^L)$ that is there exists a unitary operator $W: \mathcal{V} \to \mathcal{V}^L$ such that

$$WU(g) = U^L(g)W(g \in G)$$
$$WP(\Psi) = P^L(\Psi)W(\Psi \in C_o(M))$$

Proof. Considering that the kernel of P is invariant under translations, we grasp that $\|P(\Psi)\| = \|\Psi\|_\infty$, the supremum norm of Ψ.

Next, let us examine the Garding domain

$$\mathcal{D} = span\{U(\varphi)x | x \in \mathcal{V}, \varphi \in C_o(G)\}$$

where $U(\varphi) = \int_G \varphi(a)U(a^{-1})da$

and the Radon measure $\varphi \mapsto < P(\tau\varphi)x, y >$ for $x, y \in \mathcal{V}$, denoted $d\mu_{x,y}$, so

$$< p(\tau\varphi)x, y > = \int_G \varphi(a)d\mu_{x,y}(a)da\ (\varphi \in C_o(G)) \tag{4.1}$$

Next we show that for $x, y \in \mathcal{D}$, $d\mu_{x,y}$ is a continuous function.

Let $x, y \in M$ and $\Psi_1, \Psi_2, \varphi \in C_o(G)$.

Then,

$$\begin{aligned} | < P(\tau\varphi)u(\Psi_1)x, y > | &\leq \|\tau\varphi\|_\infty \|\Psi_1\|_\infty vol(supp\Psi_1) \cdot \|x\| \cdot \|y\| \\ &\leq c\|\Psi_1\|_\infty vol(supp\Psi_1) \cdot \|x\| \cdot \|y\| \\ &\leq c. vol(supp\Psi_1) \cdot \|x\| \cdot \|y\|. \|\varphi\|_\infty \cdot \|\Psi_1\|_\infty \end{aligned}$$

with a constant c determined by the support of $\varphi_1$so$< P(\tau\varphi)U(\varphi_1)x, y >$ defines a Radon measure $d\lambda(a, b)\ on\ G \times G$. For, if$f \in C_o(G \times G)$, set $f_b(\tau(a)) = \int_H f(\xi a, b)d\xi$.

Then we define

$$\int_{G\times G} f(a,b)d\lambda(a,b) = \int_G < P(f_b)U(b^{-1})x, y > db \tag{4.2}$$

Applying Fubini's theorem, we then compute

$$\begin{aligned} \int_G \varphi(a)d\mu_{U(\psi_1)x, U(\psi_2)y}(a) &= < P(\tau\varphi)u(\Psi_1)x, u(\Psi_2)y > \text{(by (4.1))} \\ &= \int_G \overline{\psi_2(c)} < P(\tau\varphi)U(\psi_1)x, U(c^{-1})y > dc \\ &= \int_G \overline{\psi_2(c)} < P(\tau R(c)\varphi)U(R(c)\psi_1)x, y > dc \\ &= \int_G \overline{\psi_2(c)} \int\int_{G\times G} \varphi(ac)\psi_1(bc)d\lambda(a,b)dc \quad \text{(by(4.2))} \\ &= \int_G \varphi(c) \int\int_{G\times G} \overline{\psi_2(a^{-1}c)}\psi_1(ba^{-1}c)\Delta(a^{-1})d\lambda(a,b)dc \end{aligned}$$

(Here change of variables in the c-integration has been made).

Further,

$$d\mu_{U(\psi_1)x,U(\psi_2)y}(c) = h_{U(\psi_1)x,U(\psi_2)y}(a)da$$

where

$h_{U(\psi_1)x,U(\psi_2)y}(c) = \int\int_{G\times G} \overline{\psi_2(a^{-1}c)}\psi_1(ba^{-1}c)\Delta(a^{-1})d\lambda(a,b)$ is a continuous function on G.

Thus $h_{U(g)U(\psi_1)x,U(\psi_2)y}(a)$ is continuous in (g,a) and more generally for $x,y\in\mathcal{D}$ that $h_{U(g_1)x,U(g_2)y}(a)$ is continuous in (g_1,g_2,a).

Now, we introduce a sesquilinear form on $\mathcal{D}\times\mathcal{D}$: $\beta(x,y) = h_{x,y}(e)$.

To see this, we have the following points:

$$\beta(x,x) = h_{x,x}(e) \geq 0 \tag{4.3}$$

$$\beta(U(\xi)x,U(\xi)y) = h_{U(\xi)x,U(\xi)y}(e) = \rho(\xi)\beta(x,y) \tag{4.4}$$

$$< P^L(\tau\varphi)x,y >= \int_G \varphi(a)\beta(U(a)x,U(a)y)da \tag{4.5}$$

Now, suppose $\mathcal{V}' = {}^{\mathcal{D}}/_{ker\beta}$ and $L(\xi)[x] = [\rho(\xi)^{-\frac{1}{2}}U(\xi)x]$.

Then $< L(\xi)[x],[y] >= \rho(\xi)^{-\frac{1}{2}}\beta(U(\xi)x,y)$ is continuous and L is a unitary continuous representation of $H\ in\ \mathcal{V}'$.For $x\in\mathcal{D}$, $f_x(a) = [U(a)x]$ is a continuous function on G verifying

$f_x(\xi a) = \rho(\xi)^{\frac{1}{2}}L(\xi)f_x(a)(\xi\in H, a\in G)$, again $W: x\to f_x$ extends to an isometry from $\mathcal{V}\ onto\ \mathcal{V}^L$ intertwining (U,P) and (U^L,P^L) as required.

Indeed we see that

$< P^L(\tau\varphi)f,g >= \int_G \varphi(a) < f(a),g(a) > da$ for $f,g\in F^*$ and with finite norm together with (4.5) prove that W intertwines $P$and P^L .

The uniqueness of L follows from

$$(U^L,P^L) = (U^{L'},P^{L'})$$

which is equivalent to

$$L\sim L'$$

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